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ON THE MOTION OF REMOTE ARTIFICIAL EARTH'S SATELLITES IN THE GRAVITATIONAL FIELDS OF THE EARTH AND OF THE MOON

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SUMMARY

The expressions are presented of first order perturbations of AES' orbit elements due to Earth's oblateness, and of the second harmonic of Moon's attraction, the Moon being considered as a material point moving along a circular orbit. The formulas derived are also valid for perturbations due to the Sun.

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The expressions were derived in a preceding work [1] for secular variations of remote AES in the gravitational fields of the Earth and of the Moon, whereupon perturbations from the second harmonic of the terrestrial potential and from the second and third harmonics of Moon's attraction were taken into account; it was moreover considered that the Moon is a material point moving along an elliptical orbit.

Remaining within the framework of the classical scheme proposed by Lagrange, we obtained perturbations in the Lagrange elements; it is then natural that the formulas obtained for the secular perturbations are valid for small eccentricities and inclinations.

In the current work we consider the perturbations of the second harmonic of Earth's potential and from the second harmonic of Moon's attraction, considering the latter as a material point moving along a circular orbit.

Secular and long-period perturbations of first order in the motion of the AES are obtained by way of transition to the independent variable of the the unperturbed true anomaly of the satellite.

The formulas thus derived are valid for orbits with arbitrary inclination.

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1. STATEMENT OF THE PROBLEM

Let us choose a rectangular system of coordinates Oxyz with origin at the Earth's center of masses in such a way that the plane xOy coincide with the orbit plane of the perturbing body (the Moon), and the axis Ox be directed into Moon orbit's perigee.

Then, expanding the perturbing function of the problem in series by Legendre polynomials, and limiting ourselves to second harmonics of Earth's potential and Moon's attraction, we may write

$$R = \frac{\mu_L}{2}\frac{r^2}{r_L^3}(3\cos^2\psi - 1) + \frac{1}{3}\mu_E J\frac{a_0^2}{r^3}\Big\{1 - \frac{3}{2}[N_1^2 + N_2^2 + \cos 2v\,(\cos 2\omega\,(N_2^2 - N_1^2) + 2N_1N_2\sin 2\omega) + \sin 2v\,(\sin 2\omega\,(N_1^2 - N_2^2) + 2N_1N_2\cos 2\omega)]\Big\}, \tag{1}$$

where

$$\mu_L = fm_L, \quad \mu_E = fm_E,$$
$$\cos\psi = \cos u\cos(\Omega - v_L) - \sin u\cos i\sin(\Omega - v_L),$$
$$N_1 = \sin i\cos\varepsilon + \cos i\sin\varepsilon\cos\Omega,$$
$$N_2 = \sin\varepsilon\sin\Omega, \quad u = v + \omega,$$

f is the gravitational constant, m_E and m_L are respectively the masses of the Earth and of the Moon, r and r_L are the geocentrical radii of the satellite and of the Moon, v and v_L are the true anomalies of the satellite and of the Moon, J is the obateness parameter, a_0 is the equatorial radius of the Earth, ω is the perigee longitude, i is the inclination, Ω is the longitude of satellite orbit plane, ε is the angle between the Earth's equatorial plane and Moon's orbit plane.

We shall denote the first term of expression (1), having the multiplier μ_L, by R_L, and the term containing the multiplier μ_E, by R_L. If in place of $\cos\psi$ we introduce into R_L its expression by the Kepler elements of the satellite and of the Moon, R_L will take the following form:

$$R_L = \frac{1}{8}n_1^2r^2(3\cos^2 i - 1) + \frac{3}{8}n_1^2r^2\Big\{\sin^2 i\,[\cos 2(\Omega - v_L) + \cos 2(v + \omega)] + 2\sin^4\frac{i}{2}\cos 2(v + v_L + \omega - \Omega) + 2\cos^4\frac{i}{2}\cos 2(v - v_L + \omega + \Omega)\Big\}, \tag{2}$$

where

$$n_1^2 = \frac{\mu_L}{r_L^3}.$$

If we assume the unperturbed true anomaly of the satellite for the new independent variable, the Lagrange equation for the osculating elliptical elements, valid for the computations of first order, will take the form [2]

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If we take the unperturbed true anomaly of the satellite for the new independent variable, the Lagrange equation for the osculating ellitpical elements, valid for the computations of first order, will take the form [2]:

$$
\begin{aligned}
\frac{da}{dv} &= \frac{2}{n^2 a}\left(\frac{r}{a}\right)^2 \frac{1}{\sqrt{1-e^2}} \frac{\partial R}{\partial M_0}, \\
\frac{de}{dv} &= \frac{\sqrt{1-e^2}}{n^2 a^2 e}\left(\frac{r}{a}\right)^2 \frac{\partial R}{\partial M_0} - \frac{1}{n^2 a^2 e}\left(\frac{r}{a}\right)^2 \frac{\partial R}{\partial \omega}, \\
\frac{d\omega}{\partial v} &= -\frac{\operatorname{ctg} i}{n^2 a^2 (1-e^2)}\left(\frac{r}{a}\right)^2 \frac{\partial R}{\partial i} + \frac{1}{n^2 a^2 e}\left(\frac{r}{a}\right)^2 \frac{\partial R}{\partial e}, \\
\frac{di}{dv} &= \frac{\operatorname{ctg} i}{n^2 a^2 (1-e^2)}\left(\frac{r}{a}\right)^2 \frac{\partial R}{\partial \omega} - \frac{1}{n^2 a^2 (1-e^2)\sin i}\left(\frac{r}{a}\right)^2 \frac{\partial R}{\partial \Omega}, \\
\frac{d\Omega}{dv} &= \frac{1}{n^2 a^2 (1-e^2)\sin i}\left(\frac{r}{a}\right)^2 \frac{\partial R}{\partial i}, \\
\frac{dM_0}{dv} &= -\frac{\sqrt{1-e^2}}{n^2 a^2 e}\left(\frac{r}{a}\right)^2 \frac{\partial R}{\partial e} - \frac{2}{n^2 a\sqrt{1-e^2}}\left(\frac{r}{a}\right)^2 \frac{\partial R}{\partial a}.
\end{aligned}
\qquad (3)
$$

Here $\underline{a}$ is the major semiaxis, $\underline{e}$ is the eccentricity, $\underline{n}$ is the average motion and M_0 is the mean satellite anomaly in the epoch.

The time $\underline{t}$ is linked with the unperturbed true anomaly by the following relation:

$$t - \tau = \frac{(1-e^2)^{3/2}}{n}\int_0^v \frac{dv}{(1+e\cos v)^2},$$

in which τ is the time of satellite passage through the perigee.

2. DEPENDENCE BETWEEN THE TRUE ANOMALIES OF THE MOON AND OF THE SATELLITE

For the computation of AES' and Moon's mean anomalies at a given moment of time $\underline{t}$ we resort to the well known formulas

$$M = M_0 + nt, \quad M_L = M_L^0 + n_L t, \qquad (4)$$

where $\underline{n}$ and n_L are the mean anomalistic motions of the satellite and of the Moon, and M_0 and $M_L{}^0$ are the mean anomalies in the initial epoch.

Let us denote the ratio of the mean motions $\underline{n}$ and n_L by $\underline{m}$ and $m = n_L/n$. Eliminating $\underline{t}$ from Eqs.(4), we shall obtain $M_L = \overline{M}_1 + mM$, where it is postulated $M_1 = M_L{}^0 - mM_0$.

Making use of the equation of the center and neglecting the eccentricity of the orbit of the Moon, we shall obtain

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$$v_L = M_1 + m\left(v - 2e\sin v + \frac{3}{4}e^2 \sin 2v + \dots\right) = M_1 + mv + mef(v),$$

where f(v) is a limited periodical function of v.

Neglecting in the expressions cos $2(v \pm v_L + \omega - \Omega)$ by small quantities em, we shall obtain the final expresion for the perturbing function R_L:

$$R_L = \frac{1}{8}n_1^2 r^2 (3\cos^2 i - 1) + \frac{3}{8}n_1^2 r^2 \Big\{\sin^2 i\,[\cos 2(\Omega - M_1 - mv) +$$

$$+ \cos 2(v + \omega)] + 2\sin^4 \frac{i}{2}\cos 2(v + mv + M_1 + \omega - \Omega) +$$

$$+ 2\cos^4 \frac{i}{2}\cos 2(v - mv - M_1 + \omega + \Omega)\Big\}. \qquad (5)$$

3. FORMULAS FOR PERTURBATIONS

Having computed the corresponding derivatives from R_L and R by elements and substiting theur expressions into Eq.(3), we shall obtain a system of differential equations of motion of the satellite, in which the right-hand parts are functions of the true anomaly v.

Taking advantage of the expansion of expressions of the form $\left(\frac{r}{n}\right)^n$ into Fourier series by multiples of the true anomaly v, we shall have

$$\left(\frac{r}{p}\right)^n = M_n^{(0)} + \sum_{k=1}^{\infty} 2M_n^{(k)}(e)\cos kv,$$

where

$$M_n^{(k)}(e) = \frac{1}{2\pi}\int_0^{2\pi}(1 + e\cos v)^{-n}\cos kv\,dv.$$

Integration of system (3) in the assumption that the orbit elements are replaced in the right-hand parts by constant quantities gives perturbations of first order

$$a = a_0 + \delta_1 a, \quad e = e_0 + \delta_1 e,$$

$$\dots\dots\dots\dots$$

$$M_0 = M_0^* + \delta_1 M_0,$$

where $a_0, e_0, \dots, M_0^*$ are integration constants; $\delta_1 a, \delta_1 e, \dots, \delta_1 M_0$ are perturbations of the respective elements.

When integrating system (3) we preserved the secular perturbations and also the periodical perturbations, of which the total period with respect to v is $2\pi/m$.

Such periodical perturbations will correspond to those terms in R_L, for which k = n and they will be called in the following long-period perturbations. The final expressions for the secular and long-period perturbations from the Moon for all the six elements have the form

$$\delta_1 a_L = 0,$$

$$\delta_1 e_L = \frac{3}{4}\left(\frac{n_1}{n}\right)^2\left(\frac{p}{a}\right)^4 \frac{1}{e}\delta_1 B_L,$$

$$\delta_1 \omega_L = -\cos i \cdot \delta_1 \Omega_L + \delta_1 C_L,$$

$$\delta_1 i_L = \frac{3}{4}\left(\frac{n_1}{n}\right)^2\left(\frac{p}{a}\right)^3 \Big\{-\mathrm{ctg}\, i \delta_1 B_L + \frac{1}{2}\cdot\frac{1}{m}\Big[M_4^{(0)} \sin i \cos 2(\Omega - M_1 - mv) + M_4^{(2)} \sin^2\frac{i}{2}\,\mathrm{tg}\frac{i}{2}\cos 2(mv + \omega - \Omega + M_1) + M_4^{(2)} \cos^2\frac{i}{2}\,\mathrm{ctg}\frac{i}{2}\cos 2(mv - \omega - \Omega + M_1)\Big]\Big\},$$

$$\delta_1 \Omega_L = \frac{3}{8}\left(\frac{n_1}{n}\right)^2\left(\frac{p}{a}\right)^3 \frac{1}{m}\Big\{\cos i\Big[-2mM_4^{(0)} v + 2mM_4^{(2)}\cdot v\cos 2\omega + M_4^{(0)}\sin 2(mv - \Omega + M_1)\Big] + M_4^{(2)}\Big[\sin^2\frac{i}{2}\sin 2(mv + \omega - \Omega + M_1) - \cos^2\frac{i}{2}\sin 2(mv - \omega - \Omega + M_1)\Big]\Big\},$$

$$\delta_1 M_{0L} = -\sqrt{1-e^2}\,\delta_1 C_L - \frac{1}{2}\left(\frac{n_1 p^2}{na^2}\right)^2 \frac{1}{\sqrt{1-e^2}}\Big(M_4^{(0)}[3\cos^2 i - 1]v + 3\Big\{\sin^2 i\Big[(M_4^{(2)}\cos 2\omega)v + \frac{1}{m}M_4^{(0)}\sin 2(mv + M_1 - \Omega)\Big] + \frac{M_4^{(2)}}{m}\Big[\sin^4\frac{i}{2}\sin 2(mv + \omega - \Omega + M_1) + \cos^4\frac{i}{2}\sin 2(mv - \omega - \Omega + M_1)\Big]\Big\}\Big).$$

Here

$$\delta_1 B_L = M_4^{(2)}\Big[v\sin^2 i\sin 2\omega - \frac{1}{m}\sin^4\frac{i}{2}\cos 2(mv + \omega - \Omega + M_1) + \frac{1}{m}\cos^4\frac{i}{2}\cos 2(mv - \omega - \Omega + M_1)\Big],$$

$$\delta_1 C_L = \frac{3}{8}\left(\frac{n_1}{n}\right)^2\left(\frac{p}{a}\right)^3 \frac{1}{e}\Big\{2M_3^{(1)}\left(\frac{1}{3} - \cos^2 i\right)v - (2M_3^{(1)} + M_4^{(1)} - M_4^{(3)})\Big[v\sin^2 i\cos 2\omega + \frac{1}{m}\sin^4\frac{i}{2}\sin 2(mv + \omega - \Omega + M_1) + \frac{1}{m}\cos^4\frac{i}{2}\sin 2(mv - \omega - \Omega + M_1)\Big] - \frac{1}{m}M_3^{(1)}\sin 2(mv - \Omega + M_1)\Big\}.$$

Following are the coefficients of Fourier series entering into the expressions for elements' perturbations:

Making use of the recurrent formulas for the coefficients $M_n^{(k)}$, obtained by E. P. Aksenov [3], we find $M_3^{(1)}$, $M_4^{(0)}$, $M_4^{(1)}$, $M_4^{(2)}$.

$$M_3^{(1)} = -\frac{3}{2} e (1 - e^2)^{-5/2},$$

$$M_4^{(0)} = \left(1 + \frac{3}{2} e^2\right)(1 - e^2)^{-7/2},$$

$$M_4^{(1)} = -\frac{1}{2} e (e^2 + 4)(1 - e^2)^{-7/2},$$

$$M_4^{(2)} = \frac{5}{2} e^2 (1 - e^2)^{-7/2}.$$

As to the expressions for the perturbations of AES orbit elements from the second harmonic of Earth's potential, we may obtain closed expressions relative to eccentricity as well as relative to orbit inclination, to which Kozai pointed for the first time [4].

The expressions presented below for the perturbations of elements differ from Kozai's formulas only in that the satellite's orbit elements refer to Moon's orbit plane, while Kozai took the equatorial elements. The expressions obtained by us are brought out here for the sake of uniformity and possibility of conducting the analysis of the joint influence of the Moon and Earth's oblateness on the motion of a remote AES.

$$\delta_1 a_E = J\left(\frac{a_0}{1 - e^2}\right)^2 \frac{1}{p} \delta_1 A_E,$$

$$\delta_1 e_E = \frac{1}{2ep} J a_0^2 \left(\frac{1}{p} \delta_1 A_E - \frac{1}{a} \delta_1 B_E\right),$$

$$\delta_1 \omega_E = -\cos i\, \delta_1 \Omega_E + \delta_1 C_E,$$

$$\begin{aligned} \delta_1 i_E = \frac{1}{2} J\left(\frac{a_0}{p}\right)^2 \frac{1}{\sin i} \Big[&\cos i\, \delta_1 B_E + \Big(\sin^2 i \sin^2 \varepsilon \sin 2\Omega - \\ &- \frac{1}{2} \sin 2i \sin 2\varepsilon \sin \Omega\Big)(v + e \sin v) + \frac{1}{2}(N_5 \cos 2\omega + \\ &+ N_6 \sin 2\omega)\left(e \sin v + \sin 2v + \frac{e}{3} \sin 3v\right) - \\ &- \frac{1}{2}(N_6 \cos 2\omega - N_5 \sin 2\omega)\left(e \cos v + \cos 2v + \frac{e}{3} \cos 3v\right)\Big], \end{aligned}$$

$$\begin{aligned} \delta_1 \Omega_E = -\frac{1}{4} J\left(\frac{a_0}{p}\right)^2 \frac{1}{\sin i} \Big[&2N_3 (v + e \sin v) + (N_4 \sin 2\omega - \\ &- N_3 \cos 2\omega)\left(e \sin v + \sin 2v + \frac{e}{3} \sin 3v\right) - \\ &- (N_4 \cos 2\omega + N_3 \sin 2\omega)\left(e \cos v + \cos 2v + \frac{e}{3} \cos 3v\right)\Big], \end{aligned}$$

$$\delta_1 M_0 = -\sqrt{1-e^2}\,\delta_1 C_E + 3J\frac{a_0^2}{ap\sqrt{1-e^2}}\left\{\left(\frac{2}{3}-N_1^2-N_2^2\right)(v+e\sin v)-\right.$$

$$-\frac{1}{2}\left[(N_2^2-N_1^2)\cos 2\omega + 2N_1N_2\sin 2\omega\right]\left(e\sin v+\sin 2v+\frac{e}{3}\sin 3v\right)+$$

$$\left.+\frac{1}{2}\left[(N_1^2-N_2^2)\sin 2\omega + 2N_1N_2\cos 2\omega\right]\left(e\cos v+\cos 2v+\frac{e}{3}\cos 3v\right)\right\},$$

where

$$\delta_1 A_E = 2e\left[1-\frac{3}{2}(N_1^2+N_2^2)\right]\left[\left(1+\frac{e^2}{4}\right)\cos v+\frac{e}{2}\cos 2v+\frac{e^2}{12}\cos 3v\right]-$$

$$-\left[(N_2^2-N_1^2)\cos 2\omega + 2N_1N_2\sin 2\omega\right]\left[\frac{e}{2}(3+e^2)\cos v+\right.$$

$$\left.+\left(1+\frac{3}{2}e^2\right)\cos 2v+\frac{3}{2}e\left(1+\frac{e^2}{4}\right)\cos 3v+\frac{3}{4}e^2\cos 4v+\frac{e^3}{8}\cos 5v\right]-$$

$$-\left[(N_1^2-N_2^2)\sin 2\omega + 2N_1N_2\cos 2\omega\right]\left[\frac{e}{2}(3+e^2)\sin v+\right.$$

$$\left.+\left(1+\frac{3}{2}e^2\right)\sin 2v+\frac{3}{2}e\left(1+\frac{e^2}{4}\right)\sin 3v+\frac{3}{4}e^2\sin 4v+\frac{e^3}{8}\sin 5v\right],$$

$$\delta_1 B_E = -\left[(N_1^2-N_2^2)\sin 2\omega + 2N_1N_2\cos 2\omega\right]\left(e\sin v+\sin 2v+\frac{e}{3}\sin 3v\right)+$$

$$+\left[(N_1^2-N_2^2)\cos 2\omega - 2N_1N_2\sin 2\omega\right]\left(e\cos v+\cos 2v+\frac{e}{3}\cos 3v\right),$$

$$\delta_1 C_E = J\left(\frac{a_0}{p}\right)^2\left\{\left[1-\frac{3}{2}(N_1^2+N_2^2)\right]\left[ev+\left(1+\frac{3}{4}e^2\right)\sin v+\right.\right.$$

$$\left.+\frac{e}{2}\sin 2v+\frac{e^2}{12}\sin 3v\right]-\frac{1}{4}\left[(N_2^2-N_1^2)\cos 2\omega + 2N_1N_2\sin 2\omega\right]\times$$

$$\times\left[(2e^2-1)\sin v+3e\sin 2v+\frac{1}{3}\left(7+\frac{11}{4}e^2\right)\sin 3v+\frac{3}{2}e\sin 4v+\right.$$

$$\left.+\frac{e^2}{4}\sin 5v\right]+\frac{1}{4}\left[(N_1^2-N_2^2)\sin 2\omega + 2N_1N_2\cos 2\omega\right]\times$$

$$\times\left[\left(\frac{3}{2}e^2-1\right)\cos v+3e\cos 2v+\frac{1}{3}\left(7+\frac{11}{4}e^2\right)\cos 3v+\frac{3}{2}e\cos 4v+\right.$$

$$\left.\left.+\frac{e^2}{4}\cos 5v\right]\right\}.$$

$$N_3 = \sin 2i\cos^2\varepsilon + \cos 2i\sin 2\varepsilon\cos\Omega - \sin 2i\sin^2\varepsilon\cos^2\Omega$$

$$N_4 = \cos i\sin\Omega\sin 2\varepsilon - \sin i\sin^2\varepsilon\sin 2\Omega,$$

$$N_5 = \sin^2\varepsilon\sin 2\Omega(1+\cos^2 i)+\frac{1}{2}\sin 2i\sin 2\varepsilon\sin\Omega,$$

$$N_6 = \sin 2\varepsilon\sin i\cos\Omega + 2\sin^2\varepsilon\cos 2\Omega\cos i.$$

It is possible to obtain by the above-expounded method the expressions for long-period perturbations, taking into account e_L and e_m. Such formulas have been obtained; however, the scope of the present paper does not allow their presentation here.

In conclusion it must be noted that having taken the ecliptic plane for the basic readout plane, we may write by analogy the expression for the secular and long-period perturbations from the Sun.

**** T H E E N D ****

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